An Introduction to System-theoretic Methods for Model Reduction - Part II - Interpolatory Methods

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Model and dimension reduction in uncertain and dynamic systems
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Outline

- Linear dynamical systems:

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

  - Rational interpolation problem
  - Projection-based rational interpolation
- Optimal rational interpolation
  - Optimality in the $\mathcal{H}_2$ norm
  - Iterative Rational Krylov Algorithm
- Data-driven (frequency-domain) rational interpolation
  - Loewner framework
  - Time-domain Loewner: See Peherstorfer's talk this afternoon.
- If time allows:
  - $\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{N}\,\mathbf{x}\,\mathbf{u}(t) + \mathbf{H}\,(\mathbf{x} \otimes \mathbf{x}) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$

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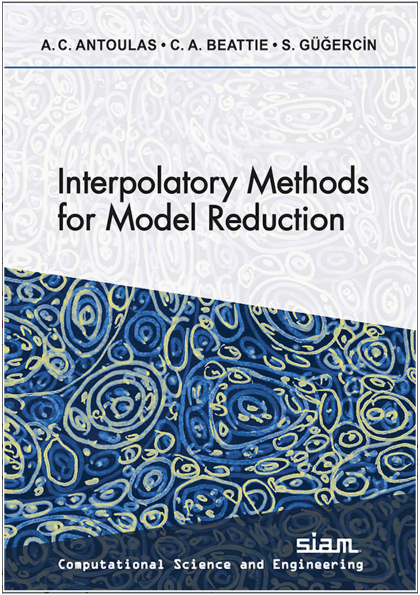
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Interpolatory Methods for Model Reduction
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Computational Science and Engineering

Indoor-air environment in a conference room

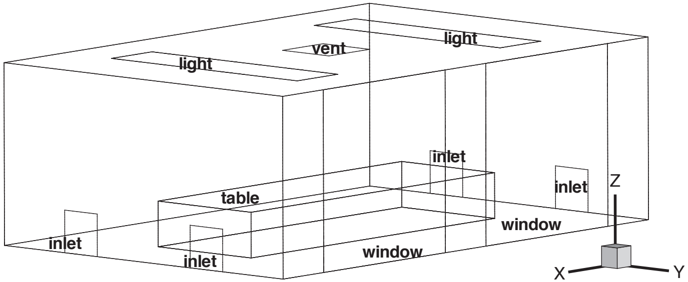

Figure: Geometry for our Indoor-air Simulation:
Example from [Borggaard/Cliff/G., 2011], research under EEBHUB

- Four inlets, one return vent
- Thermal loads: two windows, two overhead lights and occupants
- A FE model for thermal energy transfer with *frozen* velocity field $\overline{\mathbf{v}}$:

$$\frac{\partial T}{\partial t} + \bar{\mathbf{v}} \cdot \nabla T = \frac{1}{\mathrm{RePr}} \Delta T + Bu,$$

$$\implies \quad \mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

- $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}$ with $n = 202140$,
- $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{u} \in \mathbb{R}^m$ with $m = 2$ inputs (forcing)
    1. the temperature of the inflow air at all four vents, and
    2. a disturbance caused by occupancy around the conference table,
- $\mathbf{C} \in \mathbb{R}^{q \times n}$ and $\mathbf{y} \in \mathbb{R}^q$ with $q = 2$ outputs (measurements)
    1. the temperature at a sensor location on the *max x* wall,
    2. the average temperature in an occupied volume around the table,

Linear Dynamical Systems

$$\mathcal{S}: \qquad \mathbf{u}(t) \longrightarrow \boxed{\begin{aligned} \mathbf{E}\,\dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{aligned}} \longrightarrow \mathbf{y}(t)$$

- $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{q\times n}$ and $\mathbf{D} \in \mathbb{R}^{q\times m}$
- $\mathbf{x}(t) \in \mathbb{R}^n$: states, $\mathbf{u}(t) \in \mathbb{R}^m$: Input, $\mathbf{y}(t) \in \mathbb{R}^q$: Output
- State-space dimension, n, is quite large
- What is important is the mapping "$\mathbf{u} \mapsto \mathbf{y}$", NOT the complete state information $\mathbf{x}(t)$ $\Longrightarrow$ Remove the unimportant states.

Parametrized linear dynamical systems (see Beattie's talk on Feb 4)

$$\mathbf{E}(\mathsf{p})\,\dot{\mathbf{x}}(t;\mathsf{p}) = \mathbf{A}(\mathsf{p})\,\mathbf{x}(t;\mathsf{p}) + \mathbf{B}(\mathsf{p})\,\mathbf{u}(t), \quad \mathbf{y}(t;\mathsf{p}) = \mathbf{C}(\mathsf{p})\,\mathbf{x}(t;\mathsf{p}), \quad \mathsf{p} \in \mathbb{C}^{\nu}$$

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- Project the dynamics onto r-dimenisonal dimensional subspaces

$$\mathcal{S}_r : \mathbf{u}(t) \longrightarrow \boxed{\begin{aligned} \mathbf{E}_r\,\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\,\mathbf{x}_r(t) + \mathbf{B}_r\,\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\mathbf{x}_r(t) + \mathbf{D}_r\,\mathbf{u}(t) \end{aligned}} \longrightarrow \mathbf{y}_r(t) \approx \mathbf{y}(t)$$

with $\mathbf{A}_r, \mathbf{E}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$, $\mathbf{C}_r \in \mathbb{R}^{q\times r}$, and $\mathbf{D}_r \in \mathbb{R}^{q\times m}$ such that

  - $\|\mathbf{y} - \mathbf{y}_r\|$ is *small* in an appropriate norm
  - Important structural properties of $\mathcal{S}$ are preserved
  - The procedure is *computationally efficient*.

- For simplicity of notation, assume $m = q = 1$:

$$\mathbf{B} \to \mathbf{b} \in \mathbb{R}^n,\ \mathbf{C} \to \mathbf{c}^T \in \mathbb{R}^n,\ \text{and},\ \mathbf{D} \to d \in \mathbb{R} \quad\Longrightarrow\quad \mathbf{u}(t),\ \mathbf{y}(t) \in \mathbb{R}$$

For the MIMO case details, see [Antoulas/Beattie/G.,20].

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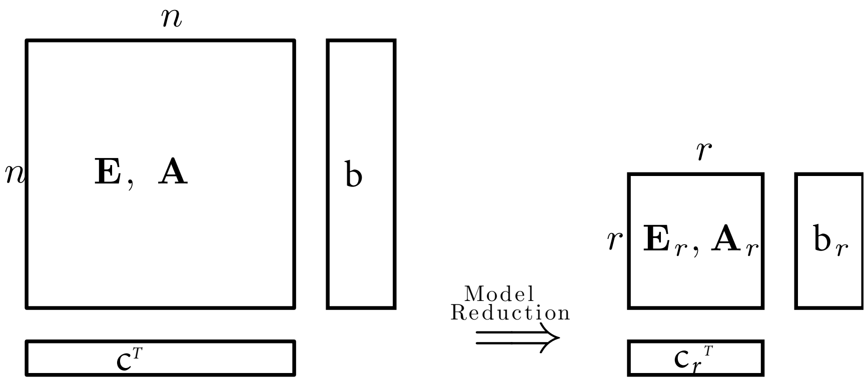

Figure: Projection-based Model Reduction

Model Reduction via Projection

- Choose $\mathcal{V}_r = \text{Range}(\mathbf{V}_r)$: the r-dimensional *right modeling subspace* (the trial subspace) where $\mathbf{V}_r \in \mathbb{R}^{n\times r}$
- and $\mathcal{W}_r = \text{Range}(\mathbf{W}_r)$, the r-dimensional *left modeling subspace* (test subspace) where $\mathbf{W}_r \in \mathbb{R}^{n\times r}$
- Approximate $\underbrace{\mathbf{x}(t)}_{n\times 1} \approx \underbrace{\mathbf{V}_r}_{n\times r}\,\underbrace{\mathbf{x}_r(t)}_{r\times 1}$ by forcing $\mathbf{x}_r(t)$ to satisfy

$$\mathbf{W}_r{}^T\left(\mathbf{E}\mathbf{V}_r\dot{\mathbf{x}}_r - \mathbf{A}\mathbf{V}_r\mathbf{x}_r - \mathbf{b}\,\mathbf{u}\right) = \mathbf{0} \quad \text{(Petrov-Galerkin)}$$

- Leads to a reduced order model:

$$\mathbf{E}_r = \underbrace{\mathbf{W}_r{}^T\mathbf{E}\mathbf{V}_r}_{r\times r},\quad \mathbf{A}_r = \underbrace{\mathbf{W}_r{}^T\mathbf{A}\mathbf{V}_r}_{r\times r},\quad \mathbf{b}_r = \underbrace{\mathbf{W}_r{}^T\mathbf{b}}_{r\times 1},\quad \mathbf{c}_r = \underbrace{\mathbf{V}_r\mathbf{c}}_{q\times r},\quad d_r = \underbrace{d}_{1\times 1}$$

Impulse Response and Transfer Functions

- $\mathcal{S}: \quad u(t) \mapsto y(t) = (\mathcal{S}u)(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)d\tau.$
- Let $\mathbf{E} = \mathbf{I}$ and $d = 0$: $h(t) = \mathbf{c}^T e^{\mathbf{A}t}\,\mathbf{b}$ (impulse response)
- $\mathcal{H}(s) = \int_{0}^{\infty} h(\tau)e^{-s\tau}d\tau = \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b} + d.$

  = Transfer function

- Take $\mathbf{E} = \mathbf{I}_2$, $\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $d = 0.$

$$h(t) = e^{-t} - e^{-2t} \quad \Longleftrightarrow \quad \mathcal{H}(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} + \frac{-1}{s+2}$$

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- Let $\hat{y}(\omega) = \mathcal{F}(y(t))$, $\hat{y}_r(\omega) = \mathcal{F}(y_r(t))$, and $\hat{u}(\omega) = \mathcal{F}(u(t))$.

$$\text{Full response:} \quad \hat{y}(\omega) = \mathcal{H}(\imath\omega)\hat{u}(\omega)$$

$$\text{Reduced order response:} \quad \hat{y}_r(\omega) = \mathcal{H}_r(\imath\omega)\hat{u}(\omega)$$

with transfer functions:

$$\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} + d \quad \text{and} \quad \mathcal{H}_r(s) = \mathbf{c}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r + d_r$$

$$y(t) \approx y_r(t) \iff h(t) \approx h_r(t) \iff \mathcal{H}(s) \approx \mathcal{H}_r(s)$$

$$h(t) = \sum_{i=1}^{n} \psi_i e^{\nu_i t} \iff \mathcal{H}(s) = \frac{\alpha_0 s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}{s^n + \beta_1 s^{n-1} + \cdots + \beta_n}$$

$$h_r(t) = \sum_{j=1}^{r} \phi_j e^{\lambda_j t} \iff \mathcal{H}_r(s) = \frac{\hat{\alpha}_0 s^r + \hat{\alpha}_1 s^{r-1} + \cdots + \hat{\alpha}_r}{s^r + \hat{\beta}_1 s^{r-1} + \cdots + \hat{\beta}_r}$$

Frequency Domain Plots

- We will illustrate the error mostly in the frequency domain.
- Amplitude Bode Plot: Draw $\|\mathcal{H}(\imath\omega\|_2$ vs $\omega \in \mathbb{R}$.
- For the previous dynamical systems, we obtain the following:

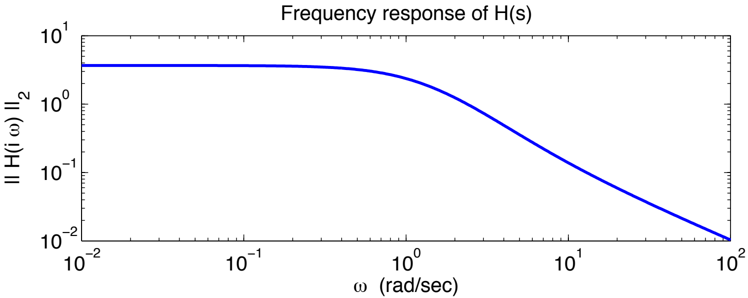


Figure: Frequency Response of $\mathcal{H}(s)$

Error measure: $\mathcal{H}_2$ Norm

- L_2 norm of $h(t)$ in time domain.
- $2-\infty$ induced norm of $\mathcal{S}$ (when $m=1$ and/or $q=1$:)

$$\|\mathfrak{H}\|_{\mathcal{H}_2} = \|h\|_{L_2} = \|\mathcal{S}\|_{2,\infty} = \sup_{u\neq 0} \frac{\|y\|_{L_\infty}}{\|u\|_{L_2}}$$

- In general (for MIMO systems)

$$\|\mathfrak{H}\|_{\mathcal{H}_2} = \left(\int_0^\infty \|\mathbf{h}(t)\|_F^2 \, dt \right)^{\frac{1}{2}} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \|\mathfrak{H}(\imath\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}$$

$$\|\mathbf{y}-\mathbf{y}_r\|_{L_\infty} \leq \|\mathfrak{H}-\mathfrak{H}_r\|_{\mathcal{H}_2} \, \|\mathbf{u}\|_{L_2}$$

How to compute the $\mathcal{H}_2$ norm:

- To have $\|\mathcal{S}\|_{\mathcal{H}_2} < \infty$, we need $d = \mathbf{0}$.
- Given $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$, let $\mathbf{P}$ be the unique solution to

  $$\mathbf{APE}^T + \mathbf{E}^T\mathbf{PA} + \mathbf{bb}^T = \mathbf{0}.$$

  Then,

  $$\|\mathcal{S}\|_{\mathcal{H}_2} = \sqrt{\mathbf{c}^T\,\mathbf{P}\,\mathbf{c}}$$

- Directly follows from the definition.
- Matlab commands: `norm(`$\mathcal{S}$`,2)`, `normh2(`$\mathcal{S}$`)`, `h2norm(`$\mathcal{S}$`)`,

Error measure: $\mathcal{H}_\infty$ Norm

- 2-2 induced norm of $\mathcal{S}$:

$$\|\mathcal{S}\|_{\mathcal{H}_\infty} = \sup_{u\neq 0} \frac{\|y\|_2}{\|u\|_2} = \sup_{u\neq 0} \frac{\|\mathcal{S}u\|_2}{\|u\|_2} = \sup_{w\in\mathbb{R}} \|\mathbf{\mathcal{H}}(\imath w)\|_2$$

- $\|\mathcal{S} - \mathcal{S}_r\|_{\mathcal{H}_\infty}$ = Worst output error $\|y(t) - y_r(t)\|_2$ for $\|u\|_2 = 1$.

$$\|y - y_r\|_{L_2} \leq \|\mathbf{\mathcal{H}} - \mathbf{\mathcal{H}}_r\|_{\mathcal{H}_\infty} \|u\|_{L_2}$$

How to compute the $\mathcal{H}_\infty$ norm:

- Let $d = 0$
- $\|\mathcal{S} - \mathcal{S}_r\|_{\mathcal{H}_\infty} \leq \gamma$ if and only if the matrix pencil

$$\lambda \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^T \end{bmatrix} - \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma}\mathbf{b}\mathbf{b}^T \\ -\frac{1}{\gamma}\mathbf{c}\mathbf{c}^T & -\mathbf{A}^T \end{bmatrix}$$

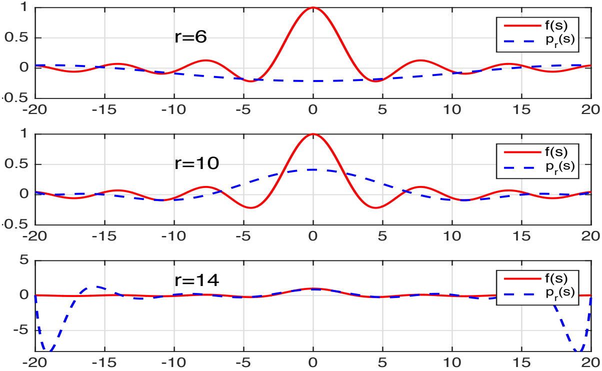
  has no purely imaginary eigenvalues.
- Computationally intensive: [Boyd/Balakrishnan,1990], [Boyd/Balakrishnan/Kabamba,1989], [Bruinsma/Steinbuch,1990], [Benner/Byers/Mehrmann/Xu,1999], [Benner/Voigt, 2012], [Benner/Voigt, 2012], [Aliyev et al., 2017],
- Matlab commands: `norm(`$\mathcal{S}$`,inf)`, `norminf(`$\mathcal{S}$`)`, `hinfnorm(`$\mathcal{S}$`)`,

Interpolating $f(s)$

- Given the interpolation nodes $\{s_i\}_{i=0}^r$, find $p_r(s)$ such that

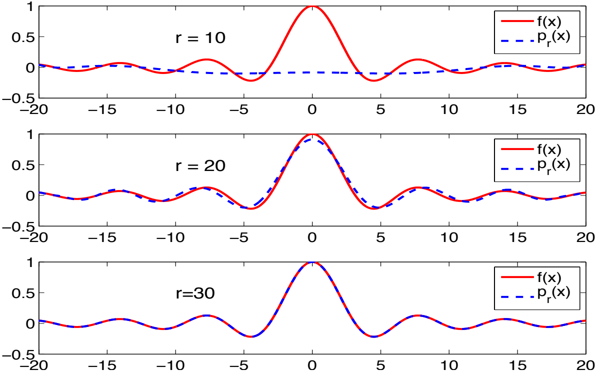
$$f(s_i) = p_r(s_i) \quad \text{for} \quad i = 0, \ldots, r.$$

- Consider $f(s) = \dfrac{\sin s}{s}$ for $s \in [-20, 20]$. Use linearly spaced nodes:



Interpolating $f(s)$

- $f(s) = \dfrac{\sin s}{s}$ for $s \in [-20, 20]$. Use Chebyshev nodes:



Model Reduction by Rational Interpolation

- Given a transfer function $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$ together with

  interpolation points: $\{\mu_i\}_{i=1}^{r} \subset \mathbb{C}$,   *interpolation points*: $\{\sigma_j\}_{j=1}^{r} \subset \mathbb{C}$

- Find a reduced model $\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$, that is a rational interpolant to $\mathcal{H}(s)$:

  $\mathcal{H}_r(\mu_i) = \mathcal{H}(\mu_i)$ for $i = 1, \cdots, r$,   and $\mathcal{H}_r(\sigma_j) = \mathcal{H}(\sigma_j)$ for $j = 1, \cdots, r$,

Interpolatory Model Reduction via Projection

- Given $\{\sigma_j\}_{j=1}^{r}$ and $\{\mu_i\}_{i=1}^{r}$, set

$$\mathbf{V}_r = \left[(\sigma_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{b}, \ \cdots \ , \ (\sigma_r \mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\right] \in \mathbb{C}^{n \times r} \text{ and}$$
$$\mathbf{W}_r = \left[(\mu_1 \mathbf{E}^T - \mathbf{A}^T)^{-1}\mathbf{c} \cdots (\mu_r \mathbf{E}^T - \mathbf{A}^T)^{-1}\mathbf{c}\ \right] \in \mathbb{C}^{n \times r}$$

- Obtain $\mathcal{H}_r(s)$ via projection as before

$$\mathbf{E}_r = {\mathbf{W}_r}^T \mathbf{E} \mathbf{V}_r \quad \mathbf{A}_r = {\mathbf{W}_r}^T \mathbf{A} \mathbf{V}_r, \quad \mathbf{b}_r = {\mathbf{W}_r}^T \mathbf{b}, \quad \mathbf{c}_r = {\mathbf{V}_r}^T \mathbf{c}, \quad d_r = d.$$

- Then

$$\begin{aligned} \mathcal{H}(\sigma_j) &= \mathcal{H}_r(\sigma_j), && \text{for } j = 1, \cdots, r, \\ \mathcal{H}(\mu_i) &= \mathcal{H}_r(\mu_i), && \text{for } i = 1, \cdots, r, \\ \mathcal{H}'(\sigma_k) &= \mathcal{H}'_r(\sigma_k) && \text{if } \sigma_k = \mu_k \end{aligned}$$

- [Skelton *et. al.*, 87], [Feldmann/Freund, 95], [Grimme, 97]

Interpolation Proof:

- Let $\mathcal{V}_r = \text{Ran}(\mathbf{V}_r)$ and $\mathcal{W}_r = \text{Ran}(\mathbf{W}_r)$. Define

$$\mathcal{P}_r(z) = \mathbf{V}_r(z\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{W}_r{}^T(z\mathbf{E} - \mathbf{A})$$

- $\mathcal{P}_r^2(z) = \mathcal{P}_r(z)$ with $\mathcal{V}_r = \text{Ran}(\mathcal{P}_r(z))$

$$\begin{aligned}\mathcal{H}(\sigma_k) - \mathcal{H}_r(\sigma_k) &= \mathbf{c}^T(\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} - \mathbf{c}_r^T(\sigma_k\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r \\ &= \mathbf{c}^T(\sigma_k\mathbf{E} - \mathbf{A})^{-1}(\mathbf{I} - \mathcal{Q}_r(\sigma_k))(\sigma_k\mathbf{E} - \mathbf{A})\Big(\mathbf{I} - \mathcal{P}_r(\sigma_k)\Big)(\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\end{aligned}$$

- Since $\mathbf{v} = (\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} \in \text{Ran}(\mathbf{V}_r) = \text{Ran}(\mathcal{P}_r(z))$:

$$\Big(\mathbf{I} - \mathcal{P}_r(\sigma_k)\Big)(\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} = \Big(\mathbf{I} - \mathcal{P}_r(\sigma_k)\Big)\mathbf{v} = \mathbf{v} - \mathcal{P}_r(\sigma_k)\mathbf{v} = \mathbf{v} - \mathbf{v} = 0.$$

$$\Longrightarrow \quad \mathcal{H}(\sigma_k) = \mathcal{H}_r(\sigma_k)$$

Interpolation Proof:

- Analogously define $\mathcal{Q}_r(z) = (z\mathbf{E} - \mathbf{A})\mathbf{V}_r(z\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{W}_r^T$
- $\mathcal{Q}_r^2(z) = \mathcal{Q}_r(z)$ with $\mathcal{W}_r^\perp$ = Ker$(\mathcal{Q}_r(z))$ = Ran $(\mathbf{I} - \mathcal{Q}_r(z))$. Then,

$$\mathcal{H}(z) - \mathcal{H}_r(z) = \mathbf{c}^T(z\mathbf{E} - \mathbf{A})^{-1}\left(\mathbf{I} - \mathcal{Q}_r(z)\right)(z\mathbf{E} - \mathbf{A})\Big(\mathbf{I} - \mathcal{P}_r(z)\Big)(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

- Evaluate at $z = \mu_k$ to obtain: $\mathcal{H}(\mu_k) = \mathcal{H}_r(\mu_k)$
- Evaluate at $z = \sigma + \varepsilon$:

$$\mathcal{H}(\sigma_i + \varepsilon) - \mathcal{H}_r(\sigma_i + \varepsilon) = \mathcal{O}(\varepsilon^2).$$

Since $\mathcal{H}(\sigma_i) = \mathcal{H}_r(\sigma_i)$,

$$\frac{1}{\varepsilon}\left(\mathcal{H}(\sigma_i + \varepsilon) - \mathcal{H}(\sigma_i)\right) - \frac{1}{\varepsilon}\left(\mathcal{H}_r(\sigma_i + \varepsilon) - \mathcal{H}_r(\sigma_i)\right) \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

Reduction from $n = 2$ to $r = 1$ ☺

- Recall the simple example

$$\mathbf{E} = \mathbf{I}_2, \quad \mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

- $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} = \dfrac{1}{s^2 + 3s + 2}$
- Choose $\sigma_1 = \mu_1 = 0$.
- $\mathbf{V}_r = (\sigma_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \qquad \mathbf{W}_r = (\mu_1 \mathbf{E} - \mathbf{A})^{-T}\mathbf{c} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$

- $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} = \dfrac{1}{s^2 + 3s + 2}$
- $\mathbf{E}_r = \mathbf{W}_r{}^T\mathbf{E}\mathbf{V}_r = 0.75, \qquad \mathbf{A}_r = \mathbf{W}_r{}^T\mathbf{A}\mathbf{V}_r = -0.5,$
- $\mathbf{b}_r = \mathbf{W}_r{}^T\mathbf{b} = 0.5, \qquad \mathbf{c}_r = \mathbf{V}_r{}^T\mathbf{c} = 0.5,$
- $\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r = \dfrac{\frac{1}{3}}{s + \frac{2}{3}}$
- $\mathcal{H}(\sigma_1) = \mathcal{H}(0) = \mathcal{H}_r(0) = 0.5 \qquad \checkmark$
- $\mathcal{H}'(\sigma_1) = \mathcal{H}_r'(0) = -0.75 \qquad \checkmark$

- Let $\mathbf{E} = \mathbf{I} \quad \Rightarrow \quad \mathbf{V}_r = \left[(\sigma_1 \mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \; \cdots, \; (\sigma_r \mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \right]$
- Then $\mathbf{V}_r$ solves

$$\mathbf{V}_r \Sigma - \mathbf{A}\mathbf{V}_r = \mathbf{b}\mathbf{e}^T,$$

where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ and $\mathbf{e} = [1, 1, \ldots, 1]^T$.

- Similarly, $\mathbf{W}_r$ solves

$$\mathbf{W}_r \mathbf{M} - \mathbf{A}^T \mathbf{W}_r = \mathbf{c}\mathbf{e}^T$$

where $\mathbf{M} = \mathrm{diag}(\mu_1, \ldots, \mu_r)$

Higher-order Interpolation

Theorem

Let $\sigma \in \mathbb{C}$ be such that both $\sigma\,\mathbf{E} - \mathbf{A}$ and $\sigma\,\mathbf{E}_r - \mathbf{A}_r$ are invertible.

(a) if $\left((\sigma\,\mathbf{E}-\mathbf{A})^{-1}\mathbf{E}\right)^{j-1}(\sigma\,\mathbf{E}-\mathbf{A})^{-1}\mathbf{b} \in \mathrm{Ran}(\mathbf{V}_r)$ *for* $j = 1,.,N$

then $\mathcal{H}^{(\ell)}(\sigma) = \mathcal{H}_r^{(\ell)}(\sigma)$ *for* $\ell = 0, 1, \ldots, N-1$

(b) if $\left((\mu\,\mathbf{E}-\mathbf{A})^{-T}\mathbf{E}^T\right)^{j-1}(\mu\,\mathbf{E}-\mathbf{A})^{-T}\mathbf{c} \in \mathrm{Ran}(\mathbf{W}_r)$ *for* $j = 1,.,M,$

then $\mathcal{H}^{(\ell)}(\mu) = \mathcal{H}_r^{(\ell)}(\mu)$ *for* $\ell = 0, 1, \ldots, M-1;$

(c) if both (a) and (b) hold, and if $\sigma = \mu$,

then $\mathcal{H}^{(\ell)}(\sigma) = \mathcal{H}_r^{(\ell)}(\sigma),$ *for* $\ell = 1, \ldots, M+N+1$

- Proof follows similarly.

How to construct interpolants with $d_r \neq d$

- $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b} \qquad \mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r + d_r$

Theorem ([Beattie/G.,09] [Mayo/Antoulas,07])

Given $\{\mu_i\}_{i=1}^r \cup \{\sigma_j\}_{j=1}^r$, *let* $\mathbf{V}_r \in \mathbb{C}^{n\times r}$ *and* $\mathbf{W}_r \in \mathbb{C}^{n\times r}$ *be as before. Let* $\mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. *For any* $d_r \in \mathbb{C}$, *define*

$$\mathbf{E}_r(s) = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r, \quad \mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r + d_r\mathbf{e}\mathbf{e}^T,$$
$$\mathbf{b}_r = \mathbf{W}_r^T\mathbf{b} - d_r\mathbf{e}, \quad \text{and} \quad \mathbf{c}_r = \mathbf{V}_r^T\mathbf{c}_r - d_r\mathbf{e}.$$

Then with $\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r + d_r$, *we have*

$$\mathcal{H}(\sigma_i) = \mathcal{H}_r(\sigma_i) \quad \textit{and} \quad \mathcal{H}(\mu_i) = \mathcal{H}_r(\mu_i) \quad \textit{for } i = 1, \ldots, r.$$

- d_r can be chosen to meet certain requirements.

$\mathcal{H}_2$ Space: The SISO Case

- $\mathcal{H}_2$: Set of scalar-valued functions, $\mathfrak{H}(z)$, with components that are analytic for z in the open right half plane, $Re(z) > 0$, such that

$$\sup_{x>0} \int_{-\infty}^{\infty} \mid \mathfrak{H}(x + \imath y) \mid^2 \, dy < \infty.$$

- $\mathcal{H}_2$ is a Hilbert space and transfer functions associated with stable finite dimensional dynamical systems are elements of $\mathcal{H}_2$.
- For stable $\mathbf{G}(s)$ and $\mathfrak{H}(s)$:

$$\langle \mathbf{G}, \ \mathfrak{H} \rangle_{\mathcal{H}_2} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{G}(\imath\omega)} \mathfrak{H}(\imath\omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{G}(-\imath\omega) \mathfrak{H}(\imath\omega) d\omega$$

- with a norm defined as

$$\|\mathbf{G}\|_{\mathcal{H}_2} = \sqrt{\langle \mathbf{G}, \ \mathbf{G} \rangle_{\mathcal{H}_2}} \stackrel{\text{def}}{=} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mid \mathbf{G}(\imath\omega) \mid^2 \, d\omega \right)^{1/2}.$$

- For simplicity, we assume $\mathcal{H}_r(s)$ has simple poles; the theory applies to the general case.

- Pole-residue expansion of $\mathcal{H}_r(s)$ of dimension-r:

$$\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r = \sum_{i=1}^{r} \frac{\phi_i}{s-\lambda_i},$$

where $\quad \lambda_i \in \mathbb{C}_-, \quad \phi_i \in \mathbb{C} \quad$ for $i = 1, \ldots, r$.

- Note that

$$\mathcal{H}_r(s) \in \text{Span}\left\{\frac{1}{s-\lambda_1}, \frac{1}{s-\lambda_2}, \ \ldots, \frac{1}{s-\lambda_r}\right\}$$

Lemma (G./Antoulas/Beattie [2008])

Suppose that $\mathbf{G}(s)$ *and* $\mathcal{H}(s) = \sum_{i=1}^{r} \frac{\varphi_i}{s - \xi_i}$ *are real, stable and suppose that* $\mathcal{H}(s)$ *has simple poles at* $\xi_1, \xi_2, \ldots \xi_r$. *Then*

$$\langle \mathbf{G}, \mathcal{H} \rangle_{\mathcal{H}_2} = \sum_{k=1}^{r} \varphi_k \mathbf{G}(-\xi_k) \quad \text{and} \quad \|\mathcal{H}\|_{\mathcal{H}_2} = \left(\sum_{k=1}^{r} \varphi_k \mathcal{H}(-\xi_k) \right)^{1/2}.$$

- Proof: Application of the Residue Theorem:

$$\langle \mathbf{G}, \mathcal{H} \rangle_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{G}(-\imath\omega)\mathcal{H}(\imath\omega)\, d\omega = \lim_{R\to\infty} \frac{1}{2\pi\imath} \int_{\Gamma_R} \mathbf{G}(-s)\mathcal{H}(s)\, ds$$

where

$$\Gamma_R = \{z \,|\, z = \imath\omega \text{ with } \omega \in [-R, R]\} \cup \left\{ z \,\middle|\, z = R\, e^{\imath\theta} \text{ with } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \right\}.$$

Pole-residue based $\mathcal{H}_2$ error expression

Theorem

Given a full-order real system, $\mathcal{H}(s)$ and a reduced model, $\mathcal{H}_r(s)$, having the form $\mathcal{H}_r(s) = \sum_{i=1}^{r} \frac{\phi_i}{s-\lambda_i}$, the $\mathcal{H}_2$ norm of the error system is given by

$$\|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_2}^2 = \|\mathcal{H}\|_{\mathcal{H}_2}^2 - 2\sum_{k=1}^{r} \phi_k \mathcal{H}(-\lambda_k) + \sum_{k,\ell=1}^{r} \frac{\phi_k \phi_\ell}{-\lambda_k - \lambda_\ell}$$

- SISO Case: [Krajewski et al.,1995], [G./Antoulas,2003]
- MIMO Case: [Beattie/G.,2008],
- Can be used in developing descent-type $\mathcal{H}_2$ optimal model reduction algorithms [Beattie/G.,2009]

Optimal $\mathcal{H}_2$ approximation

Problem

Given $\mathcal{H}(s)$, find $\mathcal{H}_r(s)$ of order r which solves: $\displaystyle\min_{degree(\mathbf{G}_r)=r} \|\mathcal{H} - \mathbf{G}_r\|_{\mathcal{H}_2}$.

- The goal is to minimize $\displaystyle\max_{t\geq 0} \|y(t) - y_r(t)\|_\infty$ for all possible unit energy inputs.
- Non-convex optimization problem. Finding a global minimum is, at best, a formidable task.
- [Wilson,1970], [Hyland/Bernstein,1985]: Sylvester-equation based optimality conditions
- Wilson [1970]: Solution is obtained by projection. But, is it an interpolatory projection?

- Pole-residue expansion of $\mathcal{H}_r(s)$ of dimension-r:

$$\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r = \sum_{i=1}^{r} \frac{\phi_i}{s-\lambda_i} \iff \mathbf{h}_r(t) = \sum_{i=1}^{r} \phi_i e^{\lambda_i t}$$

  where

$$\lambda_i \in \mathbb{C}_- \quad \text{and} \quad \phi_i \in \mathbb{C} \qquad \text{for} \quad i = 1, \ldots, r.$$

- For simplicity, we assume $\mathcal{H}_r(s)$ has simple poles; the theory applies to the general case.

- So, where is the interpolation connection?

- Pole-residue expansion of $\mathcal{H}_r(s)$ of dimension-r:

$$\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r = \sum_{i=1}^{r} \frac{\phi_i}{s-\lambda_i} \iff \mathbf{h}_r(t) = \sum_{i=1}^{r} \phi_i e^{\lambda_i t}$$

  where

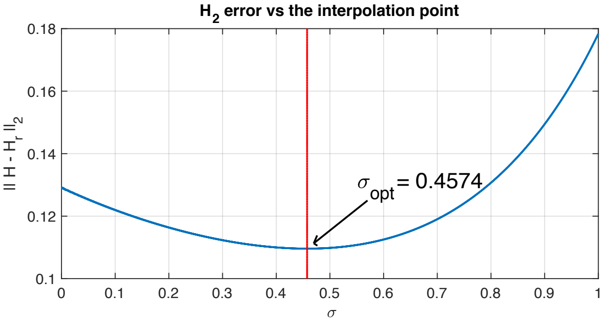
$$\lambda_i \in \mathbb{C}_- \quad \text{and} \quad \phi_i \in \mathbb{C} \qquad \text{for} \quad i = 1, \ldots, r.$$

- For simplicity, we assume $\mathcal{H}_r(s)$ has simple poles; the theory applies to the general case.

- So, where is the interpolation connection?

Searching for the optimal interpolation point

- Vary σ from $\sigma = 0$ to $\sigma = 1$ and measure the $\mathcal{H}_2$ error:

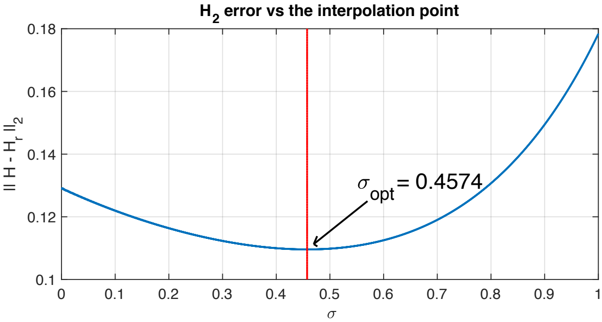


- Compute the reduced model for $\sigma_{opt} = 0.4574$, i.e., $\mathbf{V}_r = (\sigma_{opt}\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$ and $\mathbf{W}_r = (\sigma_{opt}\mathbf{E} - \mathbf{A})^{-T}\mathbf{c}$:

$$\mathcal{H}_r^{(opt)}(s) = \frac{0.2554}{s + 0.4574} \iff h_r^{(opt)}(t) = 0.2554\, e^{-0.4574\, t}$$

Searching for the optimal interpolation point

- Vary σ from $\sigma = 0$ to $\sigma = 1$ and measure the $\mathcal{H}_2$ error:



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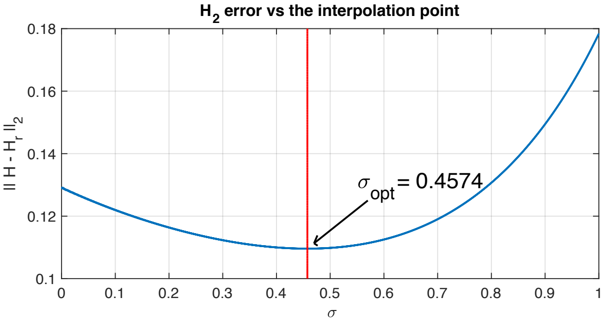


- Compute the reduced model for $\sigma_{opt} = 0.4574$, i.e., $\mathbf{V}_r = (\sigma_{opt}\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$ and $\mathbf{W}_r = (\sigma_{opt}\mathbf{E} - \mathbf{A})^{-T}\mathbf{c}$:

$$\mathcal{H}_r^{(opt)}(s) = \frac{0.2554}{s + 0.4574} \quad \Longleftrightarrow \quad h_r^{(opt)}(t) = 0.2554\, e^{-0.4574\, t}$$

Interpolatory $\mathcal{H}_2$ optimality conditions

Theorem ([Meier /Luenberger,67], [G./Antoulas/Beattie,08])

Given $\mathcal{H}(s)$, *let* $\mathcal{H}_r(s) = \sum_{i=1}^{r} \frac{\phi_i}{s-\lambda_i}$ *be the best* r^{th} *order rational approximation of* $\mathcal{H}(s)$ *with respect to the* $\mathcal{H}_2$ *norm. Then,*

$$\mathcal{H}(-\lambda_k) = \mathcal{H}_r(-\lambda_k) \text{ and } \quad \mathcal{H}'(-\lambda_k) = \mathcal{H}_r'(-\lambda_k) \quad \text{for } k = 1, 2, \ldots, r.$$

- Hermite interpolation for $\mathcal{H}_2$ optimality
- $\boxed{\text{Optimal interpolation points}: \ \sigma_i = -\lambda_i}$
- $\mathcal{H}(-\lambda_k) = \mathcal{H}_r(-\lambda_k)$ *necessary and sufficient* if λ_k are fixed.

Proof:

- $\mathcal{J} = \|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_2}^2 = \|\mathcal{H}\|_{\mathcal{H}_2}^2 - 2\sum_{k=1}^{r} \phi_k \mathcal{H}(-\lambda_k) + \sum_{k,\ell=1}^{r} \frac{\phi_k \phi_\ell}{-\lambda_k - \lambda_\ell}$
- Set the gradient to zero:

$$\frac{\partial \mathcal{J}}{\partial \phi_i} = 2(\mathcal{H}_r(-\lambda_i) - \mathcal{H}(-\lambda_i)) = 0$$
$$\frac{\partial \mathcal{J}}{\partial \lambda_i} = 2\phi_i(\mathcal{H}_r'(-\lambda_i) - \mathcal{H}'(-\lambda_i)) = 0$$

- Another interpretation

$$\langle \mathcal{H} - \mathcal{H}_r, \frac{1}{s - \lambda_i} \rangle = 0 \quad \Longrightarrow \quad \mathcal{H}(-\lambda_i) = \mathcal{H}_r(-\lambda_i)$$
$$\langle \mathcal{H} - \mathcal{H}_r, \frac{1}{(s - \lambda_i)^2} \rangle = 0 \quad \Longrightarrow \quad \mathcal{H}'(-\lambda_i) = \mathcal{H}_r'(-\lambda_i)$$

- λ_i, ϕ_i are NOT known a priori $\Longrightarrow$ Need iterative steps

An Iterative Rational Krylov Algorithm (IRKA):

Algorithm (G./Antoulas/Beattie [2008])

1. *Choose* $\{\sigma_1, \dots, \sigma_r\}$
2. $\mathbf{V}_r = \left[(\sigma_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b},\ (\sigma_2\mathbf{E}-\mathbf{A})^{-1}\mathbf{b},\ \cdots,\ (\sigma_r\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}\right]$

   $\mathbf{W}_r = \left[(\sigma_1\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c},\ (\sigma_2\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c},\ \cdots,\ (\sigma_r\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c}\right]$.
3. *while (not converged)*
   1. $\mathbf{A}_r = \mathbf{W}_r{}^T\mathbf{A}\mathbf{V}_r$, $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$
   2. $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r, \mathbf{E}_r)$.
   3. $\mathbf{V}_r = \left[(\sigma_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b},\ (\sigma_2\mathbf{E}-\mathbf{A})^{-1}\mathbf{b},\ \cdots,\ (\sigma_r\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}\right]$

      $\mathbf{W}_r = \left[(\sigma_1\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c},\ (\sigma_2\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c},\ \cdots,\ (\sigma_r\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c}\right]$.
4. $\mathbf{A}_r = \mathbf{W}_r{}^T\mathbf{A}\mathbf{V}_r$, $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$, $\mathbf{b}_r = \mathbf{W}_r{}^T\mathbf{b}$, $\mathbf{c}_r = \mathbf{V}_r{}^T\mathbf{c}$, *and* $d_r = d$.

- *Locally optimal reduced model* upon convergence. Also,

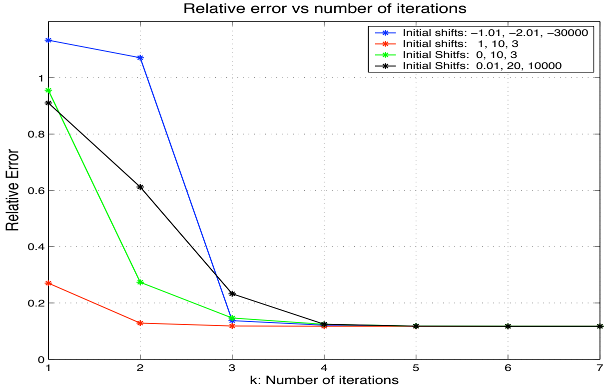
$$\mathbf{V}_r(-\Lambda) - \mathbf{A}\mathbf{V}_r = \mathbf{b}\mathbf{e}^T \quad \text{and} \quad \mathbf{W}_r(-\Lambda) - \mathbf{A}^T\mathbf{W}_r = \mathbf{c}\mathbf{e}^T$$

- In its simplest form, IRKA is a fixed point iteration. Guaranteed convergence for state-space symmetric systems [Flagg/Beattie/G.,2012]
- Newton formulation is possible [G./Antoulas/Beattie,08]
- Globally convergent descent [Beattie/G.,2009]
- Extensions
  - Structure-preservation (such as port-Hamiltonian structure): [G./Polygua/Beattie/vanderSchaft,2012], [Wyatt, 2012]
  - Data-driven implementation: [Beattie/G.,2012]
  - Extensions to $\mathcal{H}_\infty$ model reduction: [Flagg, Beattie/G.,2013]
  - Nonlinear Systems: [Benner/Breiten,2012], [Flagg/G., 2014], [Benner/Goyal/G./,2017]
  - Projected nonlinear LS framework: [Hokanson/Magruder, 2018]
- Implementation with iterative solves:
  - [Ahuja/deSturler/G./Chang, 2012], [Beattie/G./Wyatt, 2012], [Ahmad/Szyld/van Gijzen, 2017]

Small example:

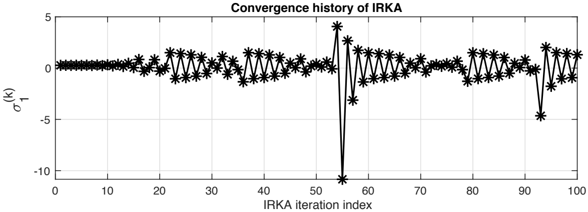
- $$\mathcal{H}(s) = \frac{2s^6 + 11.5s^5 + 57.75s^4 + 178.625s^3 + 345.5s^2 + 323.625s + 94.5}{s^7 + 10s^6 + 46s^5 + 130s^4 + 239s^3 + 280s^2 + 194s + 60}$$

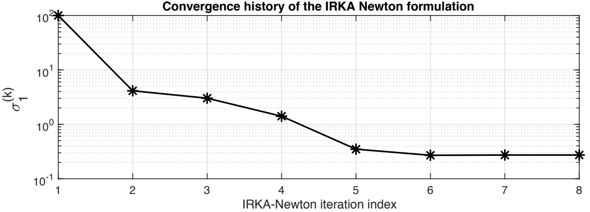
- $$\mathcal{H}_3^{(opt)}(s) = \frac{2.155s^2 + 3.343s + 33.8}{(s + 6.2217)(s + 0.61774 + \jmath 1.5628)(s + 0.61774 + \jmath 1.5628)}$$



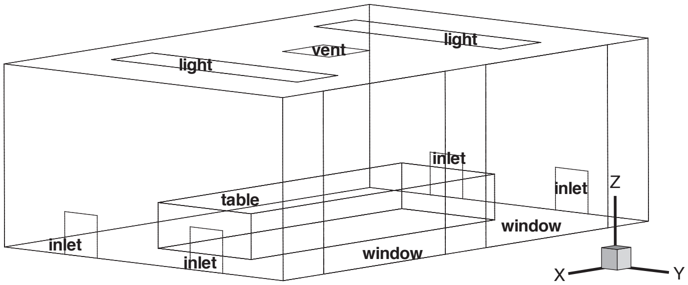
Fixed point vs Newton framework

- $\mathcal{H}(s) = \dfrac{-s^2 + (7/4)s + (5/4)}{s^3 + 2s^2 + (17/16)s + (15/32)}$, $\mathcal{H}_{\text{opt}}(s) = \dfrac{0.97197}{s + 0.27272}$
- $\dfrac{\partial \widetilde{\lambda}}{\partial \sigma} \approx 1.3728 > 1$





Indoor-air environment in a conference room



$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t),$$

- Example from [Borggaard/Cliff/G., 2011],
- Recall $n = 202140$, $m = 2$ and $p = 2$
- Reduced the order to $r = 30$ using IRKA.

- Relative errors in the subsystems by IRKA

	From Input [1]	From Input [2]
To Output [1]	6.62×10^{-3}	1.82×10^{-5}
To Output [2]	4.86×10^{-4}	5.40×10^{-7}

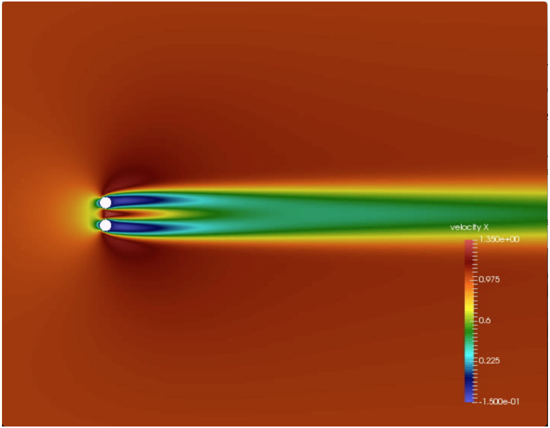
- Does IRKA pay off? How about some ad hoc selections:

	From Input [1]	From Input [2]
To Output [1]	9.19×10^{-2}	8.38×10^{-2}
To Output [2]	5.90×10^{-2}	2.22×10^{-2}

- One can keep trying different ad hoc selections but this is exactly what we want to avoid.

Wake Stabilization by Cylinder Rotation

- Joint work with Jeff Borggaard (Virginia Tech)



Wake Stabilization by Cylinder Rotation

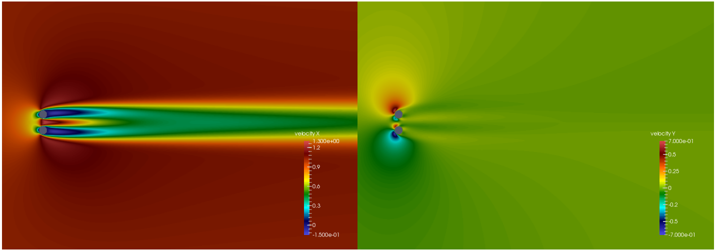


Figure: Steady-State Velocity Components at $Re_d = 60$

Goal:

Use linear feedback control to stabilize the wake behind two circular cylinders using cylinder rotation .

- [Tokumaru/Dimotakis,91], [Blackburn/Henderson,99], [Bergmann et al.,00], [Afanasiev/Hinze, 99], [Noack et al.,03], [Stoyanov, 09], [Benner/Heiland,14], ...

Optimal control problem

- Linearize the Navier-Stokes equations around the steady-state

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$$

- The LQR problem becomes: Find a control $\mathbf{u}(\cdot)$ that solves

$$\min_{\mathbf{u}} \int_0^\infty \left\{ \mathbf{y}^T(t)\mathbf{y}(t) + \alpha \|\mathbf{u}\|^2(t) \right\} dt,$$

subject to $\quad \mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$

- Instead, reduce the dimension first. Solve

$$\min_{\mathbf{u}} \int_0^\infty \left\{ \mathbf{y}_r^T(t)\mathbf{y}_r(t) + \alpha \|\mathbf{u}\|^2(t) \right\} dt,$$

subject to $\quad \mathbf{E}_r\,\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\,\mathbf{x}_r(t) + \mathbf{B}_r\,u(t), \quad \mathbf{y}_r(t) = \mathbf{C}_r\,\mathbf{x}_r(t)$

Model Reduction for the Two-cylinder Case

- $n = 126150$. We reduce the order to $r = 140$.

$$\lambda_{\text{unstable}}(\mathcal{H}(s)) : \quad 3.973912561638801 \times 10^{-2} \pm \imath\, 7.498560362688469 \times 10^{-1}$$
$$\lambda_{\text{unstable}}(\mathcal{H}_r(s)) : \quad 3.973912526082657 \times 10^{-2} \pm \imath\, 7.498560367601876 \times 10^{-1}$$

- Solve the reduced LQR problem and compute the functional gains:

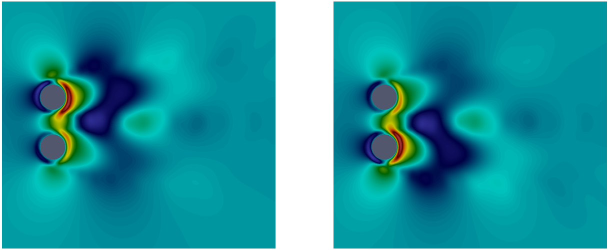
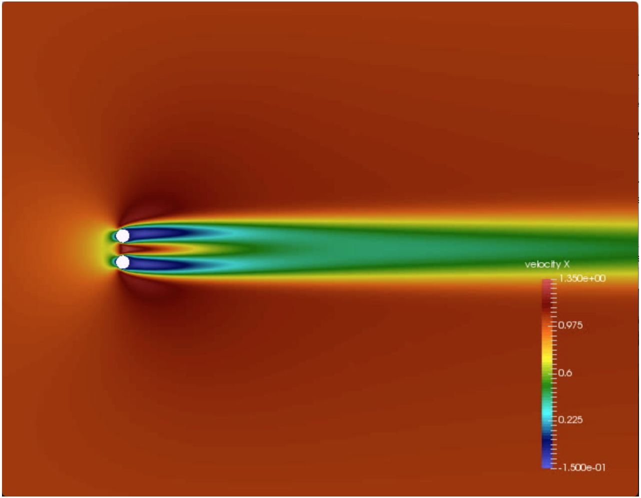
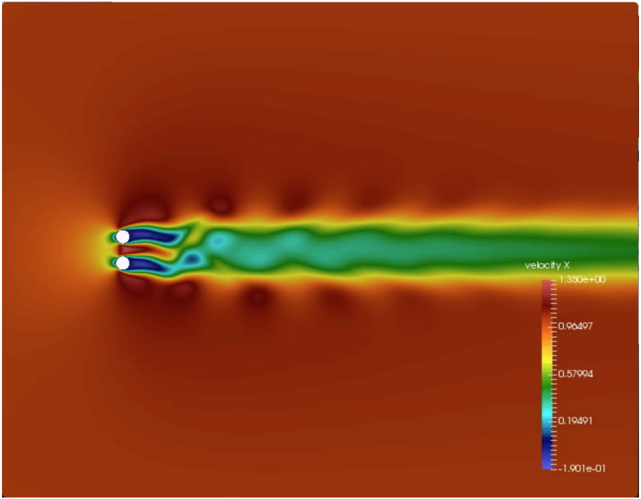


Figure: Horizontal (left) and Vertical (right) Components

$Re = 60$ Case: Open Loop Simulation



$Re = 60$ Case: Closed Loop from $t = 100$



Descent-based version: Gradient and Hessian

Theorem (Beattie./G [2009])

Let $\mathcal{H}(s)$ *and* $\mathcal{H}_r(s)$ *be given. Then, for* $i = 1, \ldots, r,$

$$\frac{\partial \mathcal{J}}{\partial \phi_i} = 2\,\left(\mathcal{H}_r(-\lambda_i) - \mathcal{H}(-\lambda_i)\right)$$

$$\frac{\partial \mathcal{J}}{\partial \lambda_i} = -2\,\phi_i\,\left(\mathcal{H}_r'(-\lambda_i) - \mathcal{H}'(-\lambda_i)\right)$$

and for $i, j = 1, \ldots, r,$

$$\frac{\partial^2 \mathcal{J}}{\partial \phi_i \partial \phi_j} = -\frac{-2}{\lambda_i + \lambda_j},$$

$$\frac{\partial^2 \mathcal{J}}{\partial \phi_i \partial \lambda_j} = -2\phi_i\,\left(\mathcal{H}_r'(-\lambda_i) - \mathcal{H}'(-\lambda_i)\right)\,\delta_{ij} + \frac{2\,\phi_j}{(\lambda_i + \lambda_j)^2}$$

$$\frac{\partial^2 \mathcal{J}}{\partial \lambda_i \partial \lambda_j} = 2\phi_i\,\left(\mathcal{H}_r''(-\lambda_i) - \mathcal{H}''(-\lambda_i)\right)\,\delta_{ij} - \frac{4\phi_i\,\phi_j}{(\lambda_i + \lambda_j)^3}$$

$\mathcal{H}_\infty$ Model Reduction Problem

- Find $\mathcal{H}_r(s) = \mathbf{c}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r + d_r$ that minimizes

$$\|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} |\mathcal{H}(\imath\omega) - \mathcal{H}_r(\imath\omega)|$$

- $d_r = 0$ forces $\mathcal{H}_r(\infty) = \mathcal{H}(\infty)$–not optimal in the $\mathcal{H}_\infty$ norm.

Theorem (Trefethen,81)

Suppose $H(s)$ is a scalar-valued transfer function associated with a SISO dynamical system. Let $\widehat{H}_r(s)$ be an optimal $\mathcal{H}_\infty$ approximation to $H(s)$ and let H_r be any r^{th} order stable approximation to $H(s)$ that interpolates $H(s)$ at $2r+1$ points in the open right half-plane. Then

$$\min_{\omega \in \mathbb{R}} |H(j\omega) - H_r(j\omega)| \leq \|H - \widehat{H}_r\|_{\mathcal{H}_\infty} \leq \|H - H_r\|_{\mathcal{H}_\infty}$$

In particular, if $|H(j\omega) - H_r(j\omega)| = \text{const}$ for all $\omega \in \mathbb{R}$ then H_r is itself an optimal $\mathcal{H}_\infty$-approximation to $G(s)$.

- $(2r+1)$ zeroes in $\mathbb{C}_+$ and nearly circular error curve $\Rightarrow$ nearly optimal approximation
- IRKA gives only $2r$-zeroes in $\mathbb{C}_+$.
- Recall: Interpolatory projection may be generalized to allow freedom in the d_r-term parameter and still preserve interpolation at the points $\{\sigma_i\}_{i=1}^{2r}$ [Mayo/Antoulas,07, Beattie/G,08]
- [Flagg/Beattie/G.,10]: Force interpolation at the $2r$ IRKA-points, and compute real-valued d_r-term that minimizes the $\mathcal{H}_\infty$ error: IHA

How to construct interpolants with $d_r \neq d$

- For optimal $\mathcal{H}_\infty$ approximants, $\lim_{s\to\infty} \mathcal{H}_r(s) \neq \lim_{s\to\infty} \mathcal{H}(s)$

Theorem ([Beattie/G.,09] [Mayo/Antoulas,07])

Given $\{\mu_i\}_{i=1}^r \cup \{\sigma_j\}_{j=1}^r$,*, let* $\mathbf{V}_r \in \mathbb{C}^{n\times r}$ *and* $\mathbf{W}_r \in \mathbb{C}^{n\times r}$ *be as before. Let*

$$\mathbf{e} = [1,\ 1, \ldots, 1]^T \in \mathbb{R}^r$$

For any $d_r \in \mathbb{C}$*, define*

$$\mathbf{E}_r(s) = \mathbf{W}_r^T \mathbf{E} \mathbf{V}_r, \quad \mathbf{A}_r = \mathbf{W}_r^T \mathbf{A} \mathbf{V}_r + d_r\, \mathbf{e}\, \mathbf{e}^T,$$

$$\mathbf{b}_r = \mathbf{W}_r^T \mathbf{b} - d_r \mathbf{e}, \quad \text{and} \quad \mathbf{c}_r = \mathbf{V}_r^T \mathbf{c} - d_r\, \mathbf{e}.$$

Then with $\mathcal{H}_r(s) = \mathbf{c}_r^T (s\mathbf{E}_r - \mathbf{A}_r)^{-1} \mathbf{b}_r + d_r$, *we have*

$$\mathcal{H}(\sigma_i) = \mathcal{H}_r(\sigma_i) \quad \textit{and} \quad \mathcal{H}(\mu_i) = \mathcal{H}_r(\mu_i) \quad \textit{for } i = 1,\ \ldots,\ r.$$

Interpolatory $\mathcal{H}_\infty$ Approximation

- Based on [Flagg/Beattie/G., 2013].
- Run IRKA to obtain $\mathcal{H}_r(s) = \mathbf{c}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$.
- Define

$$\mathcal{H}_r^d(s, d_r) = (\mathbf{c}_r - d_r\mathbf{e}^T)(s\mathbf{E}_r - (\mathbf{A}_r + d_r\mathbf{e}\mathbf{e}^T))^{-1}(\mathbf{b}_r - d_r\mathbf{e}) + d_r$$

- Solve

$$d_r^{opt} = \arg\min_{d_r} \left\| \mathcal{H} - \mathcal{H}_r^d \right\|_{\mathcal{H}_\infty}$$

- The $\mathcal{H}_\infty$ approximation via IHA is

$$\mathcal{H}_r^{opt}(s) = (\mathbf{c}_r^T - d_r^{opt}\mathbf{e}^T)(s\mathbf{E}_r - (\mathbf{A}_r + d_r^{opt}\mathbf{e}\mathbf{e}^T))^{-1}(\mathbf{b}_r - d_r^{opt}\mathbf{e}) + d_r^{opt}$$

PEEC Circuit: $n = 1434$, $r = 2$

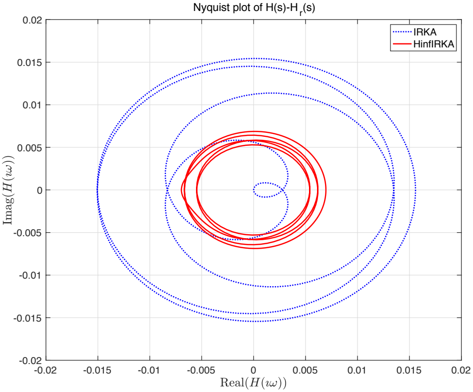


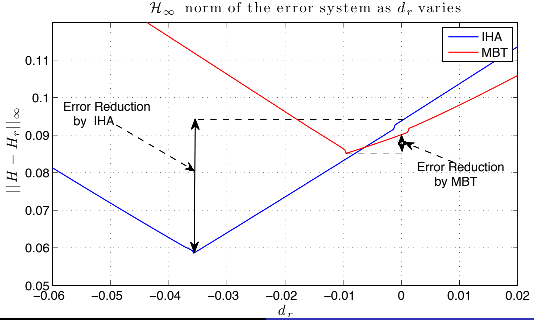
Table: Relative $\mathcal{H}_\infty$ error norms

r	IHA	BT	HNA	Lower Bound
2	4.45×10^{-3}	8.21×10^{-3}	$\mathbf{3.95 \times 10^{-3}}$	3.72×10^{-3}

CD Player Model: $n = 120$

Table: CD Player Model: Relative $\mathcal{H}_\infty$ error norms

r	IHA	BT	HNA	Lower Bound
2	3.66×10^{-1}	3.68×10^{-1}	$\mathbf{3.35 \times 10^{-1}}$	1.96×10^{-1}
4	2.14×10^{-2}	2.25×10^{-2}	$\mathbf{2.00 \times 10^{-2}}$	1.13×10^{-2}
6	$\mathbf{1.04 \times 10^{-2}}$	1.19×10^{-2}	1.23×10^{-2}	6.82×10^{-3}
8	$\mathbf{4.85 \times 10^{-3}}$	6.40×10^{-3}	5.99×10^{-3}	3.22×10^{-3}
10	$\mathbf{8.99 \times 10^{-4}}$	1.24×10^{-3}	1.08×10^{-3}	5.88×10^{-4}



Conclusions: Part I

- Uses the concept of rational interpolation and transfer function
- Optimal interpolation points in the $\mathcal{H}_2$ norm
- Extension to parametrized systems (see Beattie's talk on Feb 4)

$$\mathbf{E}(\mathsf{p})\,\dot{\mathbf{x}}(t;\mathsf{p}) = \mathbf{A}(\mathsf{p})\,\mathbf{x}(t;\mathsf{p}) + \mathbf{b}(\mathsf{p})\,u(t), \quad y(t;\mathsf{p}) = \mathbf{c}^T(\mathsf{p})\,\mathbf{x}(t;\mathsf{p}), \quad \mathsf{p} \in \mathbb{C}^\nu$$

$$\Longrightarrow \quad \mathcal{H}(s,\mathsf{p}) = \mathbf{c}^T(\mathsf{p})\,(s\mathbf{E}(\mathsf{p}) - \mathbf{A}(\mathsf{p}))^{-1}\mathbf{b}(\mathsf{p})$$

Construct $\mathcal{H}_r(s) = \mathbf{c}_r^T(\mathsf{p})\,(s\mathbf{E}_r(\mathsf{p}) - \mathbf{A}_r(\mathsf{p}))^{-1}\mathbf{b}_r(\mathsf{p})$ so that

$$\mathcal{H}(\sigma_k,\pi_j) = \mathcal{H}_r(\sigma_k,\pi_j), \qquad \frac{\partial}{\partial s}\mathcal{H}(\sigma_k,\pi_j) = \frac{\partial}{\partial s}\mathcal{H}_r(\sigma_k,\pi_j),$$
$$\nabla_{\mathsf{p}}\mathcal{H}(\sigma_k,\pi_j) = \nabla_{\mathsf{p}}\mathcal{H}_r(\sigma_k,\pi_j)$$

- Structure preserving interpolation (see Beattie's talk on Feb 4)

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A more general problem setting

- Consider the following example from [Antoulas,05]:

$$\frac{\partial T}{\partial t}(z,t) = \frac{\partial^2 T}{\partial z^2}(z,t), \quad t \geq 0, \quad z \in [0,1]$$

with the boundary conditions

$$\frac{\partial T}{\partial t}(0,t) = 0 \quad \text{and} \quad \frac{\partial T}{\partial z}(1,t) = u(t)$$

- $u(t)$ is the input function (supplied heat)
- $y(t) = T(0,t)$ is the output.
- Transfer function: $\mathcal{H}(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{\sqrt{s}\sinh\sqrt{s}} \neq \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$

- Do not assume the generic first-order structure.
- Only assume the ability to evaluate $\mathcal{H}(s)$ (and $\mathcal{H}'(s)$) at $s \in \mathbb{C}$.
- For example:
  - $\mathcal{H}(s) = \dfrac{1}{\sqrt{s}\sinh\sqrt{s}}$
  - $\mathcal{H}(s) = (s\mathbf{C}_1 + \mathbf{C}_0)(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}$
- Given the samples $\{\mathcal{H}(s_1), \mathcal{H}(s_2), \ldots, \mathcal{H}(s_N)\}$; construct:

$$\boxed{\mathcal{H}(s)} \overset{?}{\approx} \boxed{\begin{array}{c} \mathbf{E}_r\dot{\mathbf{x}} = \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t) \end{array}}$$

How to obtain the data $\{\mathcal{H}(s_1), \mathcal{H}(s_2), \ldots, \mathcal{H}(s_N)\}$?

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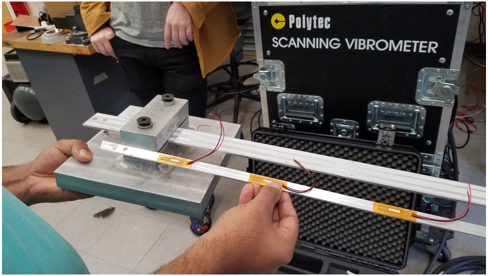
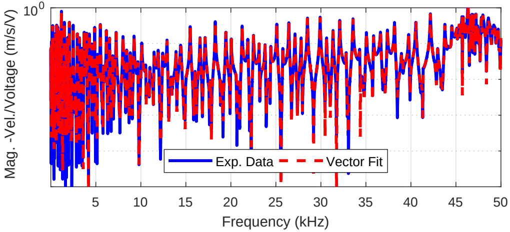
3D Laser Vibrometer (VAST LAB, Virginia Tech)





- [Malladi/Albakri/Krishnan/G./Tarazaga, 2019]

Main Ingredients: [Mayo/Antoulas (2007)]

- The *Loewner matrix*:

$$\mathbb{L}_{ij} = \frac{\mathcal{H}(\mu_i) - \mathcal{H}(\sigma_j)}{\mu_i - \sigma_j}, \quad i,j = 1,\dots,r, \quad (\mathcal{H}(s))$$

- The *shifted Loewner matrix*:

$$\mathbb{M}_{ij} = \frac{\mu_i\mathcal{H}(\mu_i) - \mathcal{H}(\sigma_j)\sigma_j}{\mu_i - \sigma_j}, \quad i,j = 1,\dots,r \quad (s\mathcal{H}(s))$$

- In addition to $\mathbb{L}$ and $\mathbb{M}$, construct the following matrices from data:

$$\mathsf{z} = \begin{bmatrix} \mathcal{H}(\mu_1) \\ \vdots \\ \mathcal{H}(\mu_r) \end{bmatrix} \qquad \mathsf{q} = \begin{bmatrix} \mathcal{H}(\sigma_1) \\ \vdots \\ \mathcal{H}(\sigma_r) \end{bmatrix}$$

Data-Driven Interpolant

Theorem (Mayo/Antoulas,2007)

Assume that $\mu_i \neq \sigma_j$ *for all* $i, j = 1, \ldots, r$. *Suppose that* $\mathbb{M} - s\,\mathbb{L}$ *is invertible for all* $s \in \{\sigma_i\} \cup \{\mu_j\}$. *Then, with*

$$\mathbf{E}_r = -\mathbb{L}, \quad \mathbf{A}_r = -\mathbb{M}, \quad \mathbf{b}_r = \mathsf{z}, \quad \mathbf{c}_r = \mathsf{q},$$

the rational function (reduced model)

$$\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r = \mathsf{q}^T(\mathbb{M} - s\,\mathbb{L})^{-1}\mathsf{z}$$

interpolates the data and furthermore is a minimal realization.

- For Hermite interpolation, choose $\sigma_i = \mu_i$ and only modify

$$\mathbb{L}_{ii} = \mathcal{H}'(\sigma_i) \quad \text{and} \quad \mathbb{M}_{ii} = [s\mathcal{H}(s)]'_{s=\sigma_i}$$

Sketch of the proof

- Assume $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$ (not necessary).
- $\mathcal{H}(\mu_i) - \mathcal{H}(\sigma_j) = (\sigma_j - \mu_i)\,\mathbf{c}^T(\mu_i\mathbf{E}-\mathbf{A})^{-1}\mathbf{E}(\sigma_j\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}.$

$$\Longrightarrow \mathbb{L} = -\mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$$

- $\mu_i\,\mathcal{H}(\mu_i) - \sigma_j\,\mathcal{H}(\sigma_j) = (\sigma_j - \mu_i)\,\mathbf{c}^T(\mu_i\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}(\sigma_j\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}.$

$$\Longrightarrow \mathbb{M} = -\mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$$

- Also $\mathsf{z} = \mathbf{W}_r^T\mathbf{b}$ and $\mathsf{q} = \mathbf{V}_r^T\mathbf{c}_r$ by definition.
  $\Rightarrow$ $\mathcal{H}_r(s) = \mathsf{q}^T(\mathbb{M} - s\,\mathbb{L})^{-1}\mathsf{z}$ is an interpolant to $\mathcal{H}(s)$.

Recall interpolatory $\mathcal{H}_2$ optimality conditions

Theorem ([Meier /Luenberger,67], [G./Antoulas/Beattie,08])

Given $\mathcal{H}(s)$, let $\mathcal{H}_r(s)$ be the best stable r^{th} order approximation of $\mathcal{H}$ with respect to the $\mathcal{H}_2$ norm. Assume $\mathcal{H}_r(s)$ has simple poles at $\hat{\lambda}_1, \hat{\lambda}_2, \ldots \hat{\lambda}_r$. Then

$$\mathcal{H}(-\hat{\lambda}_k) = \mathcal{H}_r(-\hat{\lambda}_k) \text{ and } \quad \mathcal{H}'(-\hat{\lambda}_k) = \mathcal{H}_r'(-\hat{\lambda}_k) \quad \text{for } k = 1,\, 2,\, \ldots,\, r.$$

- Hermite interpolation for $\mathcal{H}_2$ optimality
- $\boxed{\text{Optimal interpolation points}: \ \sigma_i \ = -\hat{\lambda}_i}$
- Does NOT require $\mathcal{H}(s)$ to be a rational function.
- In IRKA, replace the projection framework by the Loewner framework.

Recall IRKA

Algorithm (G./Antoulas/Beattie [2008])

1. *Choose* $\{\sigma_1, \ldots, \sigma_r\}$
2. $\mathbf{V}_r = \left[(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}, \ \cdots \ , \ (\sigma_r\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\ \right]$
   $\mathbf{W}_r = \left[(\sigma_1\,\mathbf{E}^T - \mathbf{A}^T)^{-1}\mathbf{c}^T, \ \cdots \ , \ (\sigma_r\,\mathbf{E}^T - \mathbf{A}^T)^{-1}\mathbf{c}^T\ \right].$
3. *while (not converged)*
   1. $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$, $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$
   2. $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r, \mathbf{E}_r)$. *(Reflect the current poles)*
   3. $\mathbf{V}_r = \left[(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}, \ \cdots \ , \ (\sigma_r\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\ \right]$
   4. $\mathbf{W}_r = \left[(\sigma_1\,\mathbf{E}^T - \mathbf{A}^T)^{-1}\mathbf{c}^T, \ \cdots \ , \ (\sigma_r\,\mathbf{E}^T - \mathbf{A}^T)^{-1}\mathbf{c}^T\ \right]$
4. $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$, $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$, $\mathbf{b}_r = \mathbf{W}_r^T\mathbf{b}$, *and* $\mathbf{c}_r = \mathbf{V}_r^T\mathbf{c}$, $\mathbf{D}_r = \mathbf{D}$.

- Iteratively corrected rational Hermite interpolants

Realization Independent IRKA: TF-IRKA

- Drop the need for $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$
- Only assume the ability to evaluate $\mathcal{H}(s)$ and $\mathcal{H}'(s)$

Algorithm (Realization Independent IRKA [Beattie/G., (2012)])

1. *Choose initial* $\{\sigma_i\}$ *for* $i = 1, \ldots, r$.
2. *while not converged*
   1. *Evaluate* $\mathcal{H}(\sigma_i)$ *and* $\mathcal{H}'(\sigma_i)$ *for* $i = 1, \ldots, r$.
   2. *Construct* $\mathbf{E}_r = -\mathbb{L}$, $\mathbf{A}_r = -\mathbb{M}$, $\mathbf{b}_r = \mathrm{z}$ *and* $\mathbf{c}_r = \mathrm{q}$
   3. *Construct* $\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$
   4. $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r, \mathbf{E}_r)$ *for* $i = 1, \ldots, r$
3. *Construct* $\mathcal{H}_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r = \mathrm{q}^T(\mathbb{M} - s\,\mathbb{L})^{-1}\mathrm{z}$

- Allows infinite order transfer functions !!
  e.g., $\mathcal{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A}_0 - e^{-\tau_1 s}\mathbf{A}_1 - e^{-\tau_2 s}\mathbf{A}_2)^{-1}\mathbf{B}$

Revisit: One-dimensional heat equation

- $\frac{\partial T}{\partial t}(z,t) = \frac{\partial^2 T}{\partial z^2}(z,t), \ \ \frac{\partial T}{\partial t}(0,t) = 0, \ \frac{\partial T}{\partial z}(1,t) = u(t), \ \ y(t) = T(0,t)$
- $\mathcal{H}(s) = \dfrac{1}{\sqrt{s}\sinh\sqrt{s}}$
- Apply TF-IRKA. Cost: Evaluate $\mathcal{H}(s)$ and $\mathcal{H}'(s)$!!!
- Optimal points upon convergence: $\sigma_1 = 20.9418$, $\sigma_2 = 10.8944$.

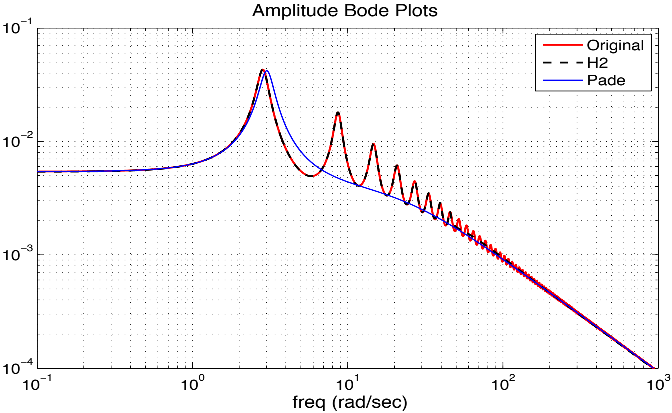
$$\mathcal{H}_r(s) = \frac{-0.9469s - 37.84}{s^2 + 31.84s + 228.1} + \frac{1}{s}$$

- $\|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_2} = 5.84 \times 10^{-3}, \ \ \|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_\infty} \approx 9.61 \times 10^{-4}$
- Balanced truncation of the discretized model:
  - $n = 1000$: $\|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_2} = 5.91 \times 10^{-3}, \ \ \|\mathcal{H} - \mathcal{H}_r\|_{\mathcal{H}_\infty} \approx 1.01 \times 10^{-3}$

Delay Example

- $\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{A}_2\mathbf{x}(t-\tau) + \mathbf{b}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{c}^T\,\mathbf{x}(t). \quad n = 1000.$
- $\mathcal{H}(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A}_1 - e^{-\tau s}\mathbf{A}_2)^{-1}\mathbf{b}.$
- $\mathcal{H}'(s) = -\mathbf{c}^T(s\mathbf{E} - \mathbf{A}_1 - e^{-\tau s}\mathbf{A}_2)^{-1}(\mathbf{E} + \tau e^{-\tau s}\mathbf{A}_2)(s\mathbf{E} - \mathbf{A}_1 - e^{-\tau s}\mathbf{A}_2)^{-1}\mathbf{b}.$
- Obtain an order $r = 20$ optimal $\mathcal{H}_2$ rational approximation directly using $\mathcal{H}(s)$ and $\mathcal{H}'(s)$
- $\mathcal{H}_r(s)$ exactly interpolates $\mathcal{H}(s)$. This will not be the case if $e^{-\tau s}$ is approximated by a rational function.
- Moreover, the rational approximation of $e^{-\tau s}$ increases the order drastically.

Delay Example



- Relative errors: TF-IRKA: 8.63×10^{-3} Pade approx: 5.40×10^{-1}
- Pade Model has dimension $N = 3000$!!!
- [Pontes Duff et al, 2015], [Pontes Duff et al, 2015]: Optimality for special delay systems.

An example on data-driven parametric modeling

- A parametrized (transfer) function/mapping: $\mathcal{H}(s,p)$:

$$\{\mathcal{H}(s_1,p_1),\mathcal{H}(s_1,p_2),\ldots,\mathcal{H}(s_N,p_M)\} \quad\Longrightarrow\quad \mathcal{H}_r(s,p)\approx\mathcal{H}(s,p)$$

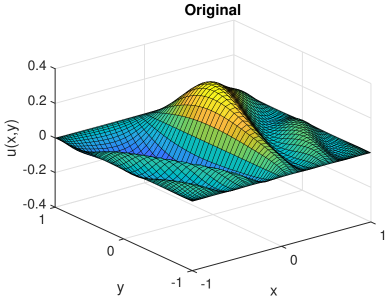
- What form of $\mathcal{H}_r(s,p)$ to choose:

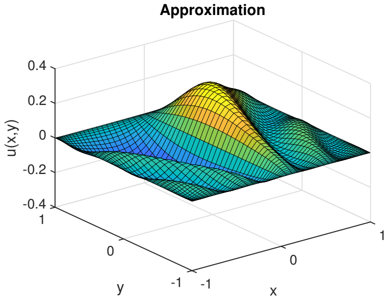
$$\mathcal{H}_r(s,p)=\sum_{i=1}^{k}\sum_{j=1}^{q}\frac{\alpha_{ij}\,h_{ij}}{(s-s_1)(p-p_j)}\Bigg/\sum_{i=1}^{k}\sum_{j=1}^{q}\frac{\alpha_{ij}}{(s-s_1)(p-p_j)}$$

- $\mathcal{H}(s_i,p_j)=\mathcal{H}_r(s_i,p_j)$ for $i=1,\ldots,k$ and $j=1,\ldots,q$
- pAAA: Pick α_{ij} to minimize a LS error in the rest of the data ([Carracedo Rodriguez/G.,2019])
- Parametric-Loewner (full interpolation): [Lefteriu/Antoulas, 2013] and [Ionita/Antoulas, 2014]

A parametrized stationary PDE ([Chen/Jiang/Narayan, 19])

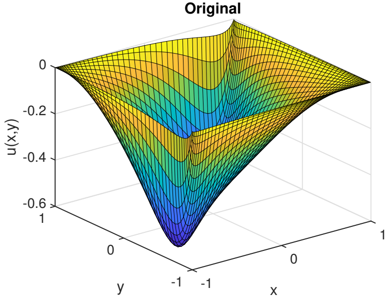
- $v_{xx} + p v_{yy} + s v = 10 \sin(8x(y-1))$ on $\Omega = [-1,1] \times [-1,1]$ with homogeneous Dirichlet boundary conditions.
- $N = M = 20$ linearly spaced samples in $[0.1, 4] \times [0, 2]$.
- pAAA results in $(k, q) = (3, 3)$. ([Carracedo Rodriguez/G.,2019])

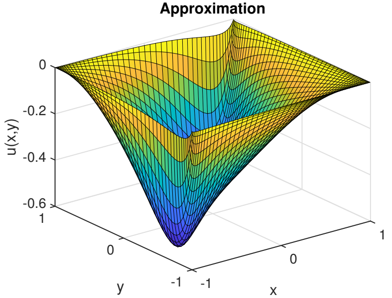




A parametrized stationary PDE ([Chen/Jiang/Narayan, 19])

- $(1+px)v_{xx} + (1+sy)v_{yy} = e^{4xy}$ on $\Omega = [-1,1] \times [-1,1]$ with homogeneous Dirichlet boundary conditions.
- $N = M = 20$ linear samples in $[-0.99, 0.99] \times [-0.99, 0.99]$.
- pAAA results in $(k,q) = (14,8)$. ([Carracedo Rodriguez/G.,2019])





Conclusions

- Interpolatory model reduction is good for you !!!
- A powerful framework for model reduction.
- Can create locally optimal reduced models effectively.
- Extended to parametrized systems.
- Data-driven formulation
- Extended to bilinear and quadratic-in-state systems.

URL: https://personal.math.vt.edu/gugercin/publications.html

1. S. Gugercin, A.C. Antoulas, and C.A. Beattie, *$\mathcal{H}_2$ model reduction for large-scale linear dynamical systems*, SIMAX, 2008.
2. C.A. Beattie and S. Gugercin, *A Trust Region Method for Optimal $\mathcal{H}_2$ Model Reduction*, Proceedings of the 48th IEEE Conference on Decision and Control, 2009.
3. G. Flagg and S. Gugercin, *Multipoint Volterra Series Interpolation and $\mathcal{H}_2$ Optimal Model Reduction of Bilinear Systems*, SIMAX, 2015.
4. A.C. Antoulas, C.A. Beattie and S. Gugercin, *Interpolatory Model Reduction of Large-scale Dynamical Systems*, Efficient Modeling and Control of Large-Scale System, Springer-Verlag, 2010.
5. J. Borggaard and S. Gugercin, *Model Reduction for DAEs with an Application to Flow Control*, Active Flow and Combustion Control 2014, Springer-Verlag, 2015.
6. C.A. Beattie and S. Gugercin, *Model Reduction by Rational Interpolation*, Model Reduction and Approximation: Theory and Algorithms, SIAM, 2017.
7. P. Benner, P. Goyal, and S. Gugercin, *$\mathcal{H}_2$-Quasi-Optimal Model Order Reduction for Quadratic-Bilinear Control Systems*, SIMAX, 2018.
8. A.C. Antoulas, C.A. Beattie, and S. Gugercin, *Interpolatory Methods for Model Reduction*, SIAM Publications, Philadelphia, PA, 2020.